Mechanics and Relativity for Mathematicians
- Wed Apr 8 2026: 15:00 - 17:00 -

Write your name and student number on **all** sheets. This exam consists of four problems, with a total of 90 points. The point distribution for each question is given in parentheses beside its title. You are allowed to bring one **handwritten** A4 page, double-sided, as a cheat sheet. Ensure that your answers are legible, well-structured, and logically presented.

Q1: Planetary Motion (5+5+10)

A satellite of mass m is in an elliptical orbit around the Earth, which has radius R_E and mass M_E. The orbit's perigee altitude is r_p and apogee altitude is r_a (perigee is the point closest to Earth, apogee the farthest). Assume that the only significant gravitational interaction is between the satellite and the Earth. Neglect the effects of the Sun, the Moon, and other planets.

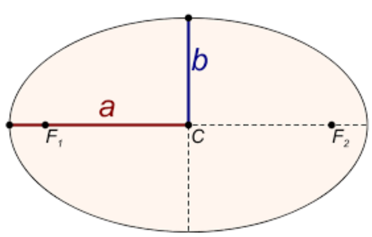
(a) The semi-major axis can be expressed in terms of r_p and r_a as $a = c\,(r_p + r_a)$, where c is a constant. What is the value of c?

(b) Along the satellite's elliptical orbit, at which point does it attain its minimum speed? Briefly explain your answer.

(c) If the orbital period of a satellite is doubled, by what factor must the size of its orbit—namely, the semi-major axis a—change? Would it increase, decrease, or remain constant? How would decreasing the mass of the satellite affect its orbital period—would it increase, decrease, or remain the same? Briefly explain your reasoning.

(a) The semi-major axis a of an ellipse is the average of the perigee and apogee distances:

$$a = \frac{r_p + r_a}{2}.$$



Comparing with $a = c\,(r_p + r_a)$, we find

$$c = \frac{1}{2}.$$

(b) The satellite's speed varies along the orbit due to conservation of mechanical energy:

$$E = \frac{1}{2}mv^2 - \frac{GM_E m}{r} = \text{constant}.$$

At perigee ($r = r_p$), the satellite is closest to Earth, so the gravitational potential energy $-GM_E m/r$ is most negative, which requires the kinetic energy $\frac{1}{2}mv^2$ to be largest in order to keep E constant. Conversely, at apogee ($r = r_a$), the satellite is farthest from Earth, so the potential energy is less negative and the kinetic energy is smaller. Therefore, the minimum speed occurs at

$$\text{apogee } (r = r_a).$$

(c) According to Kepler's third law, $T^2 \propto a^3$. If the orbital period is doubled ($T_2 = 2T_1$), then

$$\left(\frac{T_2}{T_1}\right)^2 = \left(\frac{a_2}{a_1}\right)^3 \quad \Rightarrow \quad \frac{a_2}{a_1} = 2^{2/3} \approx 1.59.$$

Thus, the semi-major axis must increase by a factor of $2^{2/3}$.

Decreasing the mass of the satellite does *not* affect the orbital period, because the period depends only on the mass of the central body (Earth) and the size of the orbit, not on the satellite's mass.

Q2: Rigid Body(20)

A uniform rod of length L and mass M is hinged at a point located a distance $L/4$ from one end. The rod is initially held at rest so that its longer end makes an angle θ above the horizontal. A small coin is placed beside the tip of the elevated end of the rod, without contact, such that the coin and the rod tip are initially at the same height. The rod and the coin are released at the same time from the rest. Assume uniform gravitational acceleration g and neglect air resistance.

Determine the critical angle θ_c at which the vertical acceleration of the rod's long end equals that of the coin immediately after release. To do this, first calculate the torque due to gravity about the hinge and use it to find the initial vertical acceleration of the rod's long end. Finally, compute the critical angle θ_c.

The hinge is at a distance $L/4$ from the short end. Then the distances from the hinge are:

$$x_{\text{CM}} = \frac{L}{2} - \frac{L}{4} = \frac{L}{4} \quad \text{(to the center of mass)}, \qquad x_{\text{long}} = \frac{3L}{4} \quad \text{(to the long end)}.$$

The torque about the hinge due to gravity is

$$\tau = x_{\text{CM}}\, Mg \sin\left(\frac{\pi}{2} + \theta\right).$$

Using the trigonometric identity $\sin(\pi/2 + \theta) = \cos\theta$, we get

$$\tau = x_{\text{CM}}\, Mg \cos\theta = \frac{L}{4} Mg \cos\theta.$$

The moment of inertia of the rod about the hinge (parallel axis theorem) is

$$I = I_{\text{CM}} + Md^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{4}\right)^2 = \frac{7}{48}ML^2.$$

The angular acceleration is then

$$\alpha = \frac{\tau}{I} = \frac{\frac{L}{4}Mg\cos\theta}{\frac{7}{48}ML^2} = \frac{12g\cos\theta}{7L}.$$

The vertical acceleration of the long end is

$$a_{\text{rod}} = x_{\text{long}}\,\alpha = \frac{3L}{4}\cdot\frac{12g\cos\theta}{7L} = \frac{36}{28}g\cos\theta = \frac{9}{7}g\cos\theta.$$

The vertical acceleration of the rod is

$$a_y = \frac{9}{7}g\cos^2\theta.$$

The critical angle θ_c is found by equating the rod tip acceleration to the coin's:

$$\frac{9}{7}g\cos^2\theta_c = g \quad\Rightarrow\quad \cos^2\theta_c = \frac{7}{9}.$$

Thus, the critical angle is

$$\theta_c = \arccos\left(\frac{\sqrt{7}}{3}\right).$$

Q3: Non-Inertial Frames (10+10)

Consider a tennis ball at rest on the ground at latitude θ on Earth (in the Northern Hemisphere), where the Earth's mass is M_E and its radius is R_E. Analyze its motion and answer the questions below in the rotating reference frame of the Earth, neglecting friction and air resistance. Use the gravitational constant G.

(a) Considering only the centrifugal force, find the angular speed ω of the Earth, in terms of the given parameters, such that the centrifugal force exactly balances gravity and the ball just loses contact with the ground. For which latitude θ would it be easiest for the ball to lift off?

(b) Now consider only the Coriolis force. A ball is thrown vertically upward with initial speed v_0 at the same latitude θ. Clearly, indicate the direction of the horizontal component of the Coriolis force responsible for the deflection during both the upward and downward motion. Where does it land relative to its launch point: east, west, north, or south?

(a) Gravity exerts a downward force given by

$$F_g = \frac{GM_E m}{R_E^2}.$$

The ball loses contact with the ground when the normal force vanishes.

In the vertical (radial) direction, the relevant forces are gravity and the component of the centrifugal force that opposes it, given by

$$F_{\text{cf}}^{(v)} = m\omega^2 R_E \cos^2\theta.$$

Therefore, the condition for the ball to lose contact is

$$m\omega^2 R_E \cos^2\theta = \frac{GM_E m}{R_E^2}.$$

Solving,

$$\omega = \sqrt{\frac{GM_E}{R_E^3 \cos^2\theta}}.$$

This is minimized when $\cos\theta$ is maximal, i.e. at the equator ($\theta = 0$). Thus, lift-off is easiest at the equator.

(b) The Coriolis force is

$$\mathbf{F}_{\text{Cor}} = -2m\,\boldsymbol{\omega} \times \mathbf{v}.$$

At latitude θ, the Earth's rotation vector $\boldsymbol{\omega}$ points toward the North Pole.

Upward motion: The velocity is upward. The Coriolis force points *west*, so the ball acquires a westward horizontal velocity.

Downward motion: The velocity is downward. The Coriolis force points *east*, opposing the westward motion.

As the ball moves upward, it is deflected to the west and acquires a westward velocity. When the ball falls back down, its vertical (radial) velocity reverses direction, and therefore the Coriolis force also reverses, now pointing east.

This eastward force reduces the westward velocity, but does not immediately cancel it. As a result, the westward motion slows down (i.e. the ball is decelerated), but the ball continues to drift westward throughout its trajectory.

Consequently, the ball ultimately lands to the west of its launch point.

Q4: Special Relativity(10+5+15)

Earth and Mars are separated by a distance of 3 light-minutes in the Earth frame. Assume both are at rest in the same inertial frame (ignore any relative motion between them).

Alice and Bob move individually toward each other with constant speed $v = 0.6c$ relative to Earth. They meet at some point between Earth and Mars (as observed in the Earth frame).

Events:

- E_1: Alice passes Earth at time $t = 0$ (Earth frame).
- E_2: Bob passes Mars at time $t = 0$ (Earth frame).

(a) When do Alice and Bob meet in the Earth frame? In Alice's frame, how long does it take for them to meet after she passes by Earth?

(b) What is Bob's speed relative to Alice?

(c) Draw a spacetime diagram in the Earth frame. Draw the worldlines of both Alice and Bob. Mark the events E_1 and E_2. Using the diagram answer the following questions: In Alice's frame, which event occurs first, E_1 or E_2?Which event occurs first in Bob's frame? Does this situation violate causality? Briefly explain your reasoning. Use the provided squared paper to draw the spacetime diagram.

(a) In the Earth frame, the distance between Earth and Mars is $L = 3$ light-minutes, and Alice and Bob move toward each other with speeds $v = 0.6c$. Each of them travels for a time t_{meet} before they meet, covering a distance of $0.6\,c\,t_{\text{meet}}$.

Thus, the total distance they need to cover is

$$3 \text{ light-min} = 0.6\,c\,t_{\text{meet}} + 0.6\,c\,t_{\text{meet}} = 1.2\,c\,t_{\text{meet}}.$$

Solving for t_{meet}, we find

$$t_{\text{meet}} = \frac{L}{1.2\,c} = \frac{3 \text{ light-min}}{1.2\,c} = 2.5 \text{ min}.$$

The Lorentz factor is

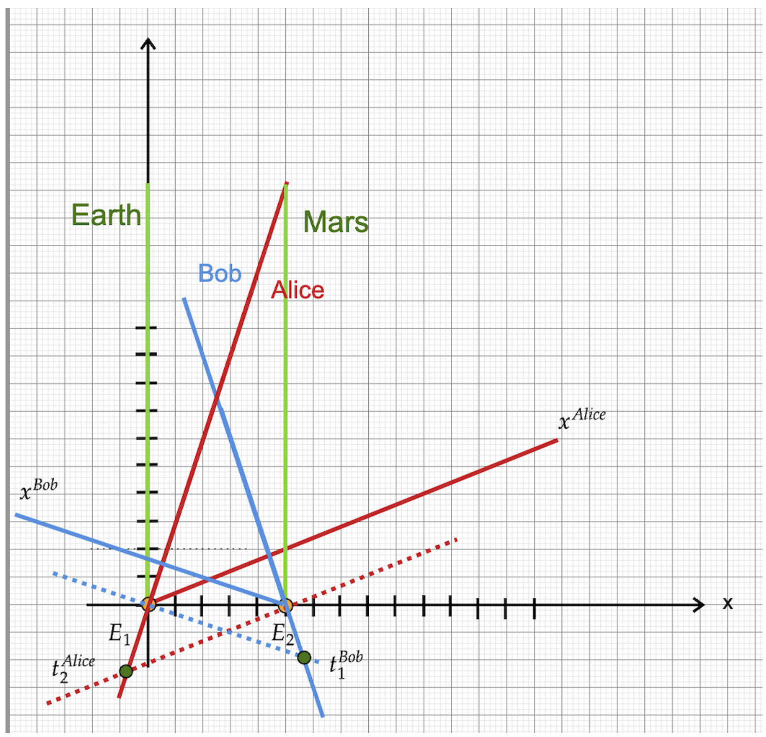
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.36}} = \frac{5}{4}.$$

In Alice's frame, the elapsed time is

$$t'_{\text{meet}} = \frac{t_{\text{meet}}}{\gamma} = \frac{2.5}{5/4} = 2.0 \text{ min.}$$

(b) Bob's speed relative to Alice is given by relativistic velocity addition:

$$u = \frac{-0.6c - 0.6c}{1 + \frac{(-0.6c)(-0.6c)}{c^2}} = -\frac{1.2c}{1.36} \approx -0.88c.$$



(c) As shown in the diagram;

- In Alice's frame, E_1 occurs before E_2.
- In Bob's frame, E_2 occurs before E_1.

The spacetime interval between E_1 and E_2 is

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 = 0 - (3 \text{ light-min})^2 < 0,$$

so the events are spacelike separated.

Therefore, different inertial frames disagree on their ordering:

This does not violate causality, because spacelike-separated events cannot influence each other; no signal traveling at or below the speed of light can connect them.